1 A quantitative assessment of the Solow model

The Solow model makes several predictions about economic growth that are testable. The steady state of the model is interesting in two respective. First, as an economy always converges towards steady state, it is a natural starting point to compare income per worker across countries. Obviously, this does not imply that each country is necessarily in its steady state but we may take it as a reasonable approximation. Moreover, most income differences that we observe across countries are highly persistent suggesting that out-of-steady-state dynamics are not of first-order importance to understand the observed income per worker differences. Second, the Solow model makes predictions about the long-run behavior of an economy that is in steady state. We have already seen several, the so called Kaldor facts, however, there are further testable implications.

Beyond the steady state, the Solow model makes strong predictions about economies that are outside their steady state, i.e., countries that are growing particularly fast (growth miracles) and countries that grow particularly slowly (growth disasters). Finally, these out-of-steady state dynamics, in turn, imply predictions about the convergence of living standards across countries.

The data that we are going to use to test these predictions come from the Penn World Tables. To measure the labor input in the Solow model, I multiply the number of working people by the average hour worked, i.e., I compute the total hours worked in an economy. Hence, exogenous differences in the labor input as well as its growth rate may arise from changes in the number of employed people or the number of hours worked per worker. Consistent with the Solow model, we will measure $1 - \alpha$ as the labor share in each country. This may vary across countries reflecting, for example, different sectoral compositions across countries. I compute the savings rate, consistent with the model, as the average investment to output ratio $s = \overline{\frac{I(t)}{Y(t)}}$. To measure education, we use the average year of schooling. The average year of schooling is obviously a very imperfect measure as it does not take into account the quality of schooling which can be quite poor in many developing economies. However, the measure has the advantage that it is widely available for many countries.

1.1 Steady state differences in income per capita

According to the Solow model, income per worker in steady state is given by

$$\left(\frac{Y(t)}{L(t)}\right)^* = \left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} \exp(\psi\, u) A(t) \tag{1}$$

Hence, we should observe that rich countries tend to have higher savings rates, population growth rates, more schooling, and/or higher technology levels.

Figure 1: Output per worker and the Solow model

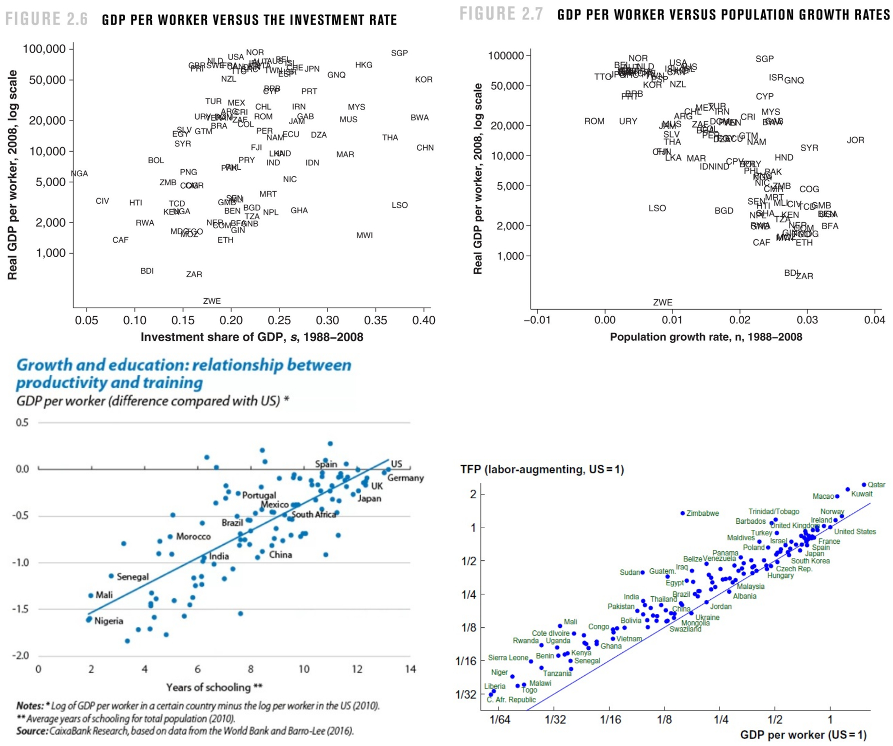


Figure 1 displays these predictions by simple unconditional relationships. Indeed, output per worker is rising in the savings rate (top-left), falling in the population growth rate (top-right), rising in education levels (bottom-left), and rising in TFP (bottom-right). These qualitative plots do not allow us to ask how well the Solow model can explain cross-country differences. To that end, Mankiw et al. (1992) assume that countries are in steady state and rewrite the above equation as

$$y(t)^* = A(0)\exp(gt)\exp(\psi\, u)\left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} \tag{2}$$

$$\ln y(t)^* = \ln A(0) + gt + \frac{\alpha}{1-\alpha}\ln s - \frac{\alpha}{1-\alpha}\ln(n+g+\delta) + \psi\, u. \tag{3}$$

Next, the authors assume that the current level of technology is randomly distributed across countries: $\ln A(0) + gt = \beta_0 + \epsilon$. Moreover, they impose a common technological growth rate and capital depreciation rate across countries: $g + \delta = 0.05$. Given these assumptions, the model can be estimated by linear OLS:

$$\ln y(t) = \beta_0 + \beta_1 \ln s + \beta_2 \ln(n + 0.05) + \beta_3 u + \epsilon(t). \tag{4}$$

Figure 2: Solow as regression model

Dependent variable: log GDP per working-age person in 1985			
Sample:	Non-oil	Intermediate	OECD
Observations:	98	75	22
CONSTANT	6.89	7.81	8.63
	(1.17)	(1.19)	(2.19)
ln(I/GDP)	0.69	0.70	0.28
	(0.13)	(0.15)	(0.39)
$\ln(n + g + \delta)$	−1.73	−1.50	−1.07
	(0.41)	(0.40)	(0.75)
ln(SCHOOL)	0.66	0.73	0.76
	(0.07)	(0.10)	(0.29)
$\bar{R}^2$	0.78	0.77	0.24

Figure 2 displays the results of the exercise. In their baseline sample of non-oil producers, variations in the savings rates, populations growth rates, and education levels explain almost 80% of the cross-sectional variation in income per worker. This large share is impressive and looks like, at first glance, a huge success for the Solow model. Importantly, the regression does not use observations on TFP differences across countries, i.e., these seem, again at first sight, not to be that important. What is more, all estimated regression coefficients have the sign predicted by the theory, i.e., positive for the savings rate and education and negative for the population growth rate and are statistically significant. Recall that the coefficient on the savings rate (population growth rate) is (-)$\frac{\alpha}{1-\alpha}$. Given the estimates, this implies $\alpha = 0.41$ ($\alpha = 0.63$). That is, the coefficient on the savings rate implies an economically sensible capital share but the estimate implied by the population growth rate is too large.

The issue of the estimated α highlights one drawback from this regression approach: the regression chooses coefficients that best fit the data, even when they are economically unreasonable. There are at least two further drawbacks. First, we have to assume that each economy is in its steady state. Second, we have to assume that technology is independent distributed from our regressors of interest. This exclusion restriction is unlikely to hold leading to serious endogenity concerns through reverse causality: countries with a high TFP are richer. In turn, as a country becomes richer, people savings abilities increase as more households move away from their subsistence level of consumption and people may start valuing education more. Moreover, as countries grow richer, the opportunity costs of having children increases leading to falling population growth rates as we observe in many developed economies today. Hence, the high R^2 from teh above regression may simply result from our regressors being systematically correlated with TFP.

An alternative approach to assess the performance of the Solow model in explaining cross country differences in income per person is called Development accounting. This approach again assumes a particular production function but, instead of choosing the parameters of the production function to best match the data, it fixes the parameters based on a reasonable estimates. In particular, Development accounting rewrites the production function according to

$$Y(t) = K(t)^{\alpha} \left(A(t)H(t)\right)^{1-\alpha} \tag{5}$$

$$Y(t)^{1-\alpha} = \left(\frac{K(t)}{Y(t)}\right)^{\alpha} \left(A(t)H(t)\right)^{1-\alpha} \tag{6}$$

$$Y(t) = \left(\frac{K(t)}{Y(t)}\right)^{\frac{\alpha}{1-\alpha}} A(t)H(t) \tag{7}$$

$$\frac{Y(t)}{L(t)} = \left(\frac{K(t)}{Y(t)}\right)^{\frac{\alpha}{1-\alpha}} A(t)\exp(\psi u). \tag{8}$$

Importantly, any cross country differences in s or n, the heart of the Solow model, should be reflected in the capital-to-output ratio. We assume $\alpha = 0.3$ and, based on micro estimates, impose $\psi = 0.1$ which is at the upper end of estimates, i.e., we give education differences a rather large chance of explaining the data. Hence, the only variable that we do not measure directly in the above equation is the TFP level $A(t)$ which will be inferred to make the equation hold. Finally, we express differences in income per country relative to some baseline country, in our case the U.S,:

$$\frac{y(t)}{y^{US}(t)} = \frac{\left(\frac{K(t)}{Y(t)}\right)^{\frac{\alpha}{1-\alpha}}}{\left(\frac{K^{US}(t)}{Y^{US}(t)}\right)^{\frac{\alpha}{1-\alpha}}} \frac{A(t)}{A^{US}(t)} \exp(\psi(u - u^{US})). \tag{9}$$

With that equation at hand, we can ask how much observed country differences in economic variables can explain differences in income per person. For example, the U.S. has around 11 years of schooling while the poorest countries have only 3. With a 10% return on schooling, we have:

$$\exp(0.1(3 - 11)) = 0.45, \tag{10}$$

i.e., differences in education can explain a 55% lower output per capita in the poorest countries. This example makes already clear that education can explain substantial differences across countries, however, it will not explain why the poorest countries have less than 10% of the income per person of the U.S. We can do a similar exercise for capital. The economic intuition is easiest to see when rewriting it in terms of capital per worker instead of the capital-to-output ratio:

$$\frac{y(t)}{y^{US}(t)} = \frac{\left(\frac{K(t)}{Y(t)}\right)^{\frac{\alpha}{1-\alpha}}}{\left(\frac{K^{US}(t)}{Y^{US}(t)}\right)^{\frac{\alpha}{1-\alpha}}} \tag{11}$$

$$\frac{y(t)}{y^{US}(t)} = \frac{\left(\frac{k(t)}{y(t)}\right)^{\frac{\alpha}{1-\alpha}}}{\left(\frac{k^{US}(t)}{y^{US}(t)}\right)^{\frac{\alpha}{1-\alpha}}} \tag{12}$$

$$\frac{y(t)}{y^{US}(t)} = \left(\frac{k(t)}{k^{US}(t)}\right)^{\alpha}. \tag{13}$$

Hence, to explain income per worker differences by a factor of 10, e.g., the difference between the U.S. and India in 2015, we need differences in capital per worker by a factor of 1000. The reason we require huge capital differences are the diminishing marginal returns to capital, i.e., making the economy more and more capital intensive creates less and less additional output. In comparison, the actual difference between the U.S. and India was a factor of 9. A yet different way to see the same point is to rewrite the decomposition in terms of marginal products:

$$\frac{y(t)}{y^{US}(t)} = \frac{\left(\frac{\alpha}{MPK(t)}\right)^{\frac{\alpha}{1-\alpha}}}{\left(\frac{\alpha}{MPK^{US}(t)}\right)^{\frac{\alpha}{1-\alpha}}} \tag{14}$$

$$\frac{y(t)}{y^{US}(t)} = \left(\frac{MPK^{US}(t)}{MPK(t)}\right)^{\frac{1-\alpha}{\alpha}}. \tag{15}$$

Hence, to explain that the U.S. is ten times richer than India in 2015, we require that the marginal product of capital is 100 times higher in India when $\alpha = \frac{1}{3}$. Such huge discrepancies in marginal products are impassible, Surely, with such huge returns on capital in India, world investors would start investing in India.

Figure 3 displays the contribution of the different variables on income per worker differences relative to the U.S. It shows that differences in the capital-to-output ratio, indeed explain very little in the differences of income per person across countries. The largest gap it can explain relative to the U.S. are 19% for Kenya, yet, Kenya has just 3% of the income per person of the U.S. This finding casts serious doubt on the regression results of Mankiw et al. (1992): It seems implausible that differences in savings rates and population growth rates can explain 80% of the variation in income per person if capital-to-output ratios are so similar across countries. Instead, and consistent with our discussion above, it seems much more likely that the regression suffers from endogeneity of the regressors. Consistent with the reverse-causality arguments made above, the figure shows that the vast majority of cross-country differences in income per person are explained by technological differences. From the perspective of the Solow model, this result is disappointing: the factor that the model takes as exogenous, i.e., does not explain, is the factor that explains most differences in income per person across countries highlighting that we require a theory that will endogenize productivity differences across countries.

Figure 3: Development accounting

	GDP per worker, y	Capital/GDP $(K/Y)^{\alpha/(1-\alpha)}$	Human capital, h	TFP	Share due to TFP
United States	1.000	1.000	1.000	1.000	–
Hong Kong	0.854	1.086	0.833	0.944	48.9%
Singapore	0.845	1.105	0.764	1.001	45.8%
France	0.790	1.184	0.840	0.795	55.6%
Germany	0.740	1.078	0.918	0.748	57.0%
United Kingdom	0.733	1.015	0.780	0.925	46.1%
Japan	0.683	1.218	0.903	0.620	63.9%
South Korea	0.598	1.146	0.925	0.564	65.3%
Argentina	0.376	1.109	0.779	0.435	66.5%
Mexico	0.338	0.931	0.760	0.477	59.7%
Botswana	0.236	1.034	0.786	0.291	73.7%
South Africa	0.225	0.877	0.731	0.351	64.6%
Brazil	0.183	1.084	0.676	0.250	74.5%
Thailand	0.154	1.125	0.667	0.206	78.5%
China	0.136	1.137	0.713	0.168	82.9%
Indonesia	0.096	1.014	0.575	0.165	77.9%
India	0.096	0.827	0.533	0.217	67.0%
Kenya	0.037	0.819	0.618	0.073	87.3%
Malawi	0.021	1.107	0.507	0.038	93.6%
Average	0.212	0.979	0.705	0.307	63.8%
1/Average	4.720	1.021	1.418	3.260	69.2%

1.2 Steady state growth

The Solow model allows us to characterize the growth behavior of a country in its steady state. We have already seen that the model is consistent with the so called Kaldor facts for the U.S. Here, we consider further ways to look at U.S. national account data following the framework proposed by Solow (1957). The framework is very similar to Development accounting by starting from a production function. However, instead of comparing across countries, we study the time series behavior of a single country:

$$Y(t) = K(t)^{\alpha} \left(A(t)H(t)\right)^{1-\alpha} \tag{16}$$

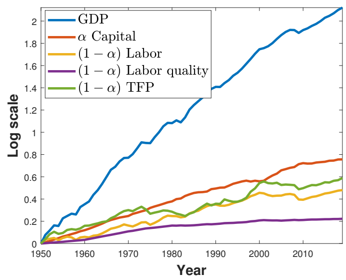
$$H(t) = L(t)\exp(\psi\, u(t)) \tag{17}$$

$$\Rightarrow \frac{\dot{Y}(t)}{Y(t)} = \alpha\frac{\dot{K}(t)}{K(t)} + (1-\alpha)\left[\frac{\dot{A}(t)}{A(t)} + \frac{\dot{L}(t)}{L(t)} + \psi\frac{\partial u(t)}{\partial t}\right]. \tag{18}$$

The equation simply says that output growth must result from capital growth, productivity growth, labor growth, or changes in the education system. The equation raises the question on how to measure the contribution coming from a changing labor quality, $\psi\frac{\partial u(t)}{\partial t}$. Growth accounting takes a Neo-classical view to measure it. That is, it assumes that each worker is paid his/her marginal product. Hence, when a college worker earns twice as much as a high-school dropout, it must be that he/she is twice as productive. Given this assumption, all we need to measure are changes in the average wages by education groups for each period and changes in education shares.

Figure 4 decomposes the output growth in the U.S. into these components. As a measure of changes in the education system, I use a measure for the quality of the workforce. This measure basically groups workers into education shares each year and estimates relative wages for the different education shares. When the share of high-wage workers rises, the measure identifies this as a rise in the quality of the workforce. The figure shows that capital accumulation is the number one contributor to output growth in the U.S. In contrast, the decomposition suggests that the increasing workforce quality has contributed relatively little to output growth over time. Note, output growth has slow down in the U.S. (and many other developed economies) around 1970. The decomposition highlights that this slowdown is almost entirely due to a slowdown in TFP growth.

Figure 4: Output growth in the U.S.



Instead of total output, we can also look at output per capita:

$$y(t) = \frac{Y(t)}{L(t)} = \left(\frac{K(t)}{L(t)}\right)^{\alpha} (A(t)\exp(\psi\, u(t)))^{1-\alpha} \tag{19}$$

$$\frac{\dot{y}(t)}{y(t)} = \alpha\frac{\dot{k}(t)}{k(t)} + (1-\alpha)\left[\frac{\dot{A}(t)}{A(t)} + \psi\frac{\partial u(t)}{\partial t}\right]. \tag{20}$$

The intuition is again very simple. Output per worker grows either because capital per worker is growing (sometimes refereed to as capital deepening), changes in the education system, or technology is growing.

Figure 5: Output per worker growth in the U.S.

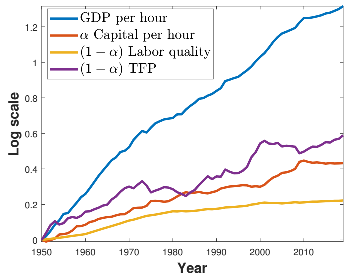


Figure 5 displays the results for the decomposition of changes in output per hour in the U.S. Productivity growth and a growth in the capital to labor ratio are the prime contributors to the growth in output per worker. One has to keep in mind that we study here a a statistical decomposition, and one needs to be careful with the interpretation. For example, a counterfactual question like: *by how much would output per worker have grown if the growth rate in technology would have been zero*, may be quite misleading. This is because our Solow model teaches us that a fundamental reason for capital per worker to grow in steady state is technological progress, i.e., if technological growth would have been absent, capital per worker would also have grown by less. Note, capital per hour is growing on a log scale by 1.07 from the beginning to the end of the sample period. The log change in technology is slightly lower, 0.99. One may be tempted to conclude a slight failure of the model as in the steady state of the standard Solow model, $g_k = g$. However, this misses the fact that labor quality is no longer constant. With labor quality rising over time, we have

$$k(t)^* = \left(\frac{s}{n+g+\delta}\right)^{\frac{1}{1-\alpha}} A(t)\exp(\psi\, u(t)) \tag{21}$$

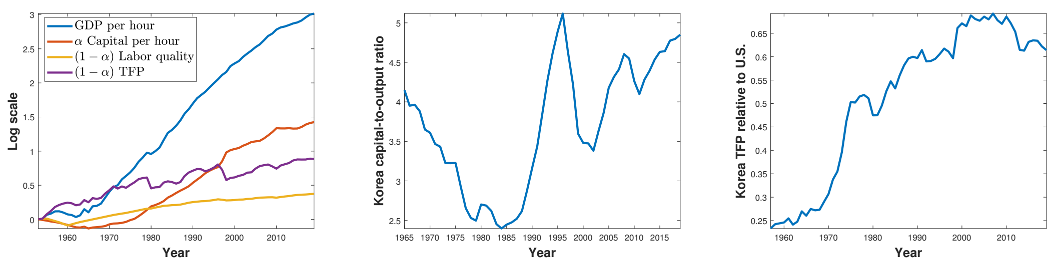
$$\left(\frac{\dot{k}(t)}{k(t)}\right)^* = \frac{\dot{A}(t)}{A(t)} + \psi\frac{\partial u(t)}{\partial t} > g. \tag{22}$$

1.3 Convergence to steady state

The Solow model also makes predictions about the behavior of economies that are outside their steady state. A country that is below its steady state should display rapid capital accumulation and, as a result, rapid output growth. In contrast, a country that is above its steady state should see its output per worker decline and its capital per worker decline even faster. Moreover, along the transition path, the absolute growth rates should slow down as the economy gets closer to its steady state. Hence, the Solow model provides a compelling theoretical framework for growth miracles and growth disasters that we can study in the data.

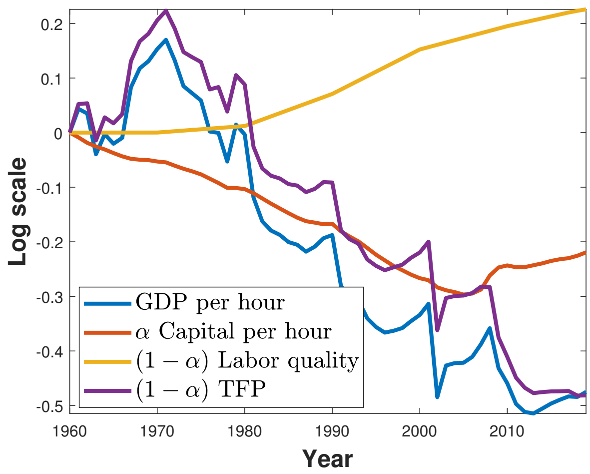
Figure 6: Growth South Korea

(a) Output per worker growth (b) Capital-to-output ratio (c) Productivity



South Korea is an example for the former as the left panel in Figure 6 shows. Between 1953 and 2019, output per worker increased by 3.02 on a log scale (compared to 1.32 in the U.S.). Consistent with the prediction of the Solow model, the growth rate of output per worker is falling over time suggesting that Korea is converging to a new steady state. On first inspection, also the growth rate of capital-to-labor appears promising. On a log scale, it grew by 2.95, much quicker than the 1.07 in the U.S. However, the key prediction of the Solow model out of steady state concerns the capital-to-output ratio and, as just discussed, the output per worker ratio grew just as much implying that the capital-to-output ratio has stayed relatively over time in Korea as the center panel shows. In fact, in Korea's fastest growth period during the late 1960s and early 1970s, even the capital-to-labor ratio changed very little. Hence, what explains the rapid growth in output per worker in Korea? The answer is TFP growth as the right panel in the figure shows. Between 1953 and 2019, Korea has closed its TFP gap relative to the U.S. from 24% to 60%.

Figure 7: Growth Madagascar

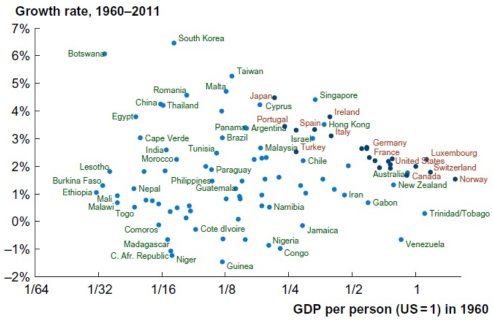


Finally, Figure 7 shows an example of a growth disaster. After some initial output pee hour growth during the 1960s, output per hour has been on a close to linear decline in Madagascar ever since then. Can we understand this behavior as an economy converging to a new steady state as in the Solow model? The answer is rather no. During the period of rising output per worker, the capital to output ratio was actually falling. Ever since, during the rapid decline in output per hour, the capital-to-output ratio was constant. Instead, a growth disaster like Madagascar mostly results from falling TFP over time highlighting once again the need to develop a theory that can explain endogenously different TFP growth rates.

1.4 Convergence in living standards

A question economists have studied extensively is whether living standards across countries converge over time, something known as absolute convergence. Figure 8 shows that the answer is generally *no*. Countries that were poor relative to the U.S. in 1960 (the x-axis) did not grow systematically faster (y-axis) than countries that were as rich as the U.S. in 1960. Obviously some poor countries did grow quickly, Botswana, South Korea, Taiwan, and Malta being prime examples, and we refer to these countries usually as growth miracles. However, just as many poor countries in the 1960s did poorly subsequently like Madagascar, Niger, Central African Republic, and Guinea, which we usually refer as the growth disasters.

Figure 8: Convergence in living standards over time



On first thought, the absence of no convergence may contradict the Solow model: Poor countries tend to have a low capital stock meaning the marginal product of capital is high which should lead to rapid capital accumulation. In contrast, the marginal product of capital is much lower in rich countries leading to relatively slow growth. However, on second thought, recall that the Solow model is fully consistent with different steady states, i.e., permanent differences, of the marginal product of capital. Only conditional on having the same production function, savings rate, population growth rate, technology, and education level should economies converge to the same steady state:

$$y(t)^* = \left(\frac{Y(t)}{L(t)}\right)^* = \left(\underbrace{\frac{s}{n+g+\delta}}_{\left(\frac{K}{Y}\right)^*}\right)^{\frac{\alpha}{1-\alpha}} A(t)\exp(\psi\, u) \tag{23}$$

Hence, the Solow model can explain permanent differences in income per person and marginal products of capital. Put differently, the Solow model predicts only what is known as conditional convergence: conditional on having the same steady state, two economies should converge. As we have already seen above, countries vastly differ in their savings rates, population growth rates, education, and technology and, hence, we should not expect absolute convergence. Baumol (1986) pointed out that countries forming the *OECD* share relatively similar socio-economic structures and, hence, we might expect them to have similar steady states. Figure 8 highlights these countries in red and shows that this indeed the case. Countries that were poor relative to the U.S. in 1960 did grow much faster in subsequent years.

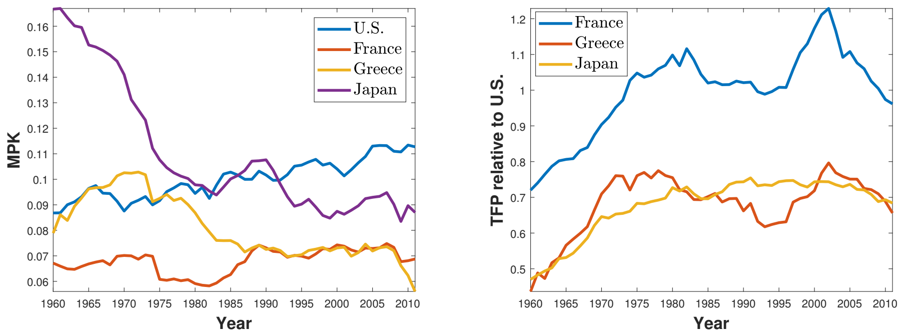
Hence, is it time to finally conclude some success for the Solow model? We may not want to be that quick. The Solow model predicts that this convergence occurs because the initially poorer countries accumulate rapidly capital leading to a convergence in capital-to-output ratios and marginal products of capital. In the data, we can compute the latter as

$$MPK(t) = \frac{\alpha}{\frac{K(t)}{Y(t)}}. \tag{24}$$

Figure 9: Convergence in the OECD

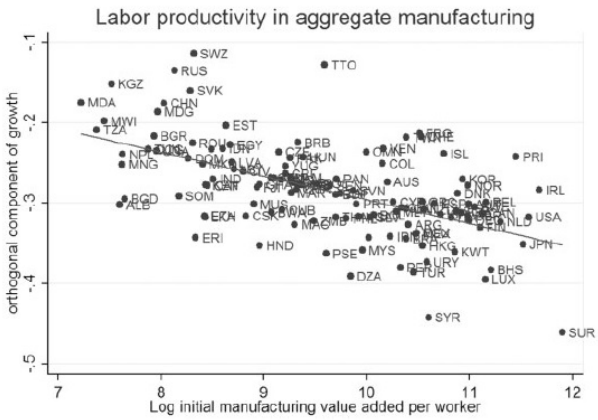
(a) MPK

(b) Productivity



The left panel of Figure 9 displays the marginal products for countries at different income-per-worker levels in 1960. Japan is the country qualitatively most consistent with the prediction of the Solow model. Between 1960 and 1980, a period of rapid income-per-worker convergence, the marginal products of capital converged. However, the model cannot explain why the marginal product of capital kept falling in Japan afterward despite Japan remaining poorer than the U.S. The model is inconsistent with Greece and France converging over time to the same steady state as the U.S. Neither started out with a marginal products of capital above the U.S. level. In fact, both are below those of the U.S for most of the sample period. Why did we nevertheless observe income-per-worker converge in those countries over time? The right panel of the figure shows that the answer is again convergence in productivities. Productivity convergence came mostly to a halt in the 1980s but so did output-per-hour convergence.

Figure 10: Convergence in manufacturing



Source: Rodrik (2013)

So far, our analysis concentrates at a country as a whole. Considering different sectors, manufacturing stands out as the sector that is most globalized with many large multinational companies operating across different countries. Hence, in manufacturing, we may expect technology to grow faster in poorer countries. Moreover, capital formation does not depend so much on local conditions such as savings rates and population growth rates. Hence, we would expect to see more convergence in manufacturing. Figure 10, taken from Rodrik (2013), shows exactly that. Different from an economy as a who;e, we observe absolute convergence across countries in manufacturing output per worker.

References

Baumol, W. J. (1986): "Productivity growth, convergence, and welfare: what the long-run data show," *The american economic review*, 1072–1085.

Mankiw, N. G., D. Romer, and D. N. Weil (1992): "A contribution to the empirics of economic growth," *The quarterly journal of economics*, 107, 407–437.

Rodrik, D. (2013): "Unconditional convergence in manufacturing," *The quarterly journal of economics*, 128, 165–204.

Solow, R. M. (1957): "Technical change and the aggregate production function," *The review of Economics and Statistics*, 312–320.